Primary decomposition

For this section, assume R is Noetherian and $M \neq 0$ is a finitely generated R-module.

Typically primary decompositions are defined for ideals of rings, but we will define it more generally for submodules.

Def: A submodule $N \subseteq M$ is <u>primary</u> if $\operatorname{Ass}(M/N)$ has just one element P. In this case, we say N is <u>P-primary</u>.

M is <u>coprimary</u> if $0 \subseteq M$ is primary. i.e. if $\operatorname{Ass} M$ consists of a single element.

By recalling how $\operatorname{Ass} M$ behaves in short exact sequences, we obtain the following:

Prop: If $P \subseteq R$ is prime, $N_1, \ldots, N_t \subseteq M$ R-modules, then if each N_i is P-primary in M, $\bigcap_i N_i$ is P-primary.

Pf: By induction, we can assume $t = 2$.

Then $M/(N_1 \cap N_2) \hookrightarrow M/N_1 \oplus M/N_2$ so

$$\operatorname{Ass}(M/(N_1 \cap N_2)) \subseteq \operatorname{Ass}(M/N_1 \oplus M/N_2) \subseteq \operatorname{Ass}(M/N_1) \cup \operatorname{Ass}(M/N_2) = \{P\}$$

Since $\mathrm{Ass}\left(M/(N_1 \cap N_2)\right) \neq \emptyset$, $N_1 \cap N_2$ is P-primary. $\square$

Using our big theorem about associated primes, we get the following characterization of coprimary modules:

Prop: Let $P \subseteq R$ be prime. The following are equivalent:

a.) M is P-coprimary.

b.) P is minimal over $\mathrm{ann}\, M$ and every elt not in P is a nonzerodivisor on M.

c.) A power of P annihilates M, and every element not in P is a NZD on M.

Note: If $M = R/I$ for some nonzero ideal I, this says I is P-primary if a power of P is in I and $\forall\ r,s \in R$ s.t. $rs \in I$ and $r \notin P \Rightarrow s \in I$.
This is the classical definition of P-primary.

$\longrightarrow$ e.g. (x^2, y) is not (x)-primary, but it's (x,y)-primary.

Pf of prop:

a.) $\Rightarrow$ b.): P is the only associated prime of M, so it must be minimal over $\mathrm{ann}\, M$, and $P = \{\text{zero divisors on } M\} \cup \{0\}$.

b.) $\Rightarrow$ c.): Let $P' = PR_P$. If $(P')^n$ annihilates M_P, then $\frac{rm}{1} = 0\ \forall\ m \in M, r \in P \Rightarrow urm = 0$ for some $u \notin P$, but u is a NZD, so r annihilates M.

Thus, we just need to show $(P')^n$ annihilates M_p.

P' is minimal over the ideal gen. by $\operatorname{ann} M = \operatorname{ann} M_p$

$\Rightarrow P' = \operatorname{rad}(\operatorname{ann} M_p)$. $\Rightarrow$ For generators $x_1, \dots, x_m$ of P', $\exists$ k_i s.t. $x_i^{k_i} \in \operatorname{ann} M_p$ $\Rightarrow$ $\exists$ $n >> 0$ s.t. $(P')^n \subseteq \operatorname{ann} M_p$.

c.) $\Rightarrow$ a.): since $P^n \subseteq \operatorname{ann} M$, $P \subseteq \operatorname{rad}(\operatorname{ann} M)$.

Thus, it must be minimal among primes over $\operatorname{ann} M$, so it's an associated prime.

Since every elt outside P is a NZD on M, all associated primes are in P, so M is P-coprimary. $\square$

Note: Part b.) tells us that M is P-coprimary $\Leftrightarrow$ P is minimal over $\operatorname{ann} M$ and M injects into M_p.

If M is any module and P minimal over $\operatorname{ann} M$, then $M' = \ker(M \to M_p)$ is P-primary since M/M' injects into $M_p = (M/M')_p$ $\left(0 \to M' \to M \to M_p \text{ exact} \Rightarrow 0 \to M'_p \to M_p \xrightarrow{\cong} M_p \text{ exact} \right)$

In this case M' is the <u>P-primary component</u> of 0 in M.

Ex: $I = (x^2 y) \subseteq k[x,y]$

Let $M = k[x,y]/I$. Then the minimal primes in $k[x,y]$ over

$\operatorname{ann} M = I$ are (x) and (y)

$$\ker(M \to M_{(x)}) = \left\{ \frac{r}{u} \;\middle|\; vr = 0 \text{ for some } v \notin (x) \right\}$$
$$= (x^2).$$

$\ker(M \to M_{(y)}) = (y)$, and $(x^2y) = (y) \cap (x^2)$

Ex: $I = (x^2, xy) \subsetneq k[x,y]$. $M = k[x,y]/I$. The only minimal prime over the annihilator is (x), and

$\ker(M \to M_{(x)}) = (x)$. However $I \neq (x)$.

Note: Part b.) of the theorem says I is P-primary
$\Rightarrow P$ is min'l over $I \Rightarrow \operatorname{rad} I \subseteq P$.

Part c.) $\Rightarrow P^n \subseteq I \Rightarrow P \subseteq \operatorname{rad} I$, so I P-primary $\Rightarrow P = \operatorname{rad} I$

However, the above example shows the converse is not true.
i.e. I is not primary just because its radical is prime:

$$\operatorname{rad}(x^2, xy) = (x), \text{ but } \operatorname{Ass}\left(R/(x^2, xy)\right) = \{(x), (x,y)\}.$$

The idea w/ primary decomposition is to write an arbitrary submodule M' of M as the intersection of primary submodules.

Thm: Let R be a Noetherian ring, and let M be a finitely generated R-module. Any proper submodule $M' \subset M$ is the intersection of finitely many primary submodules.

If $P_1, \dots, P_n$ are prime and $M' = \bigcap_{i=1}^{n} M_i$, w/ M_i P_i-primary, then

a.) Every associated prime of M/M' occurs among the P_i.

b.) If the intersection is <u>irredundant</u> (i.e. no M_i can be dropped), then the P_i are precisely the associated primes of M/M'.

c.) If the intersection is <u>minimal</u> (i.e. no intersection w/ fewer terms, then each associated prime of M/M' is equal to P_i for exactly one index i.

(In this case, if P_i is minimal over the annihilator of M/M', then M_i is the P_i-primary component of 0 in M'.)

Pf: First we prove a slightly finer decomposition.

A submodule $N \subset M$ is <u>irreducible</u> if N is not the intersection of two strictly larger submodules.

Every submodule of M can be expressed as the intersection of finitely many irreducible submodules by ACC,

so we can write $M' = \bigcap_i M_i$ w/ each M_i irreducible, called an <u>irreducible decomposition</u>.

If M_i is <u>not</u> primary, then $\exists$ P and Q distinct associated primes of M/M_i. Thus

M/P and M/Q inject into M/M_i.

The annihilator of every element of M/P is P and of M/Q is Q, so their images intersect in 0. Thus, 0 is reducible, so taking preimages in M, M_i is reducible.

Thus, each M_i is primary, so this is actually a primary decomposition.

To show a.) - c.), we factor out M' and assume $M'=0$.

a.) Suppose $0 = \bigcap M_i$ is a primary decomposition.
Then the map

$$M = M/\bigcap M_i \to \oplus M/M_i \quad \text{is an injection}$$

(since $m \mapsto 0 \iff m \in \bigcap M_i \iff m=0$.)

Thus $\operatorname{Ass} M \subseteq \bigcup \operatorname{Ass} M/M_i = \{P_i\}$.

b.) Now suppose that for each j, $\bigcap_{i\neq j} M_i \neq 0$.

Then let $A = \bigcap_{i \neq j} M_i$ and $B = M_j$, so $A \cap B = 0$.

Then $A = A/A\cap B \cong A+B/B \subseteq M/B = M/M_j$.

M_j is P_j-primary, so M/M_j is P_j-coprimary,

so A is as well. But $\operatorname{Ass} A \subseteq \operatorname{Ass} M$, so $P_j \in \operatorname{Ass} M$.

c.) Now suppose the decomposition is minimal. The intersection of P-primary submodules is P-primary, so the P_i must be distinct. By b.), we've shown the first statement.

For the second statement, assume P_i is minimal over the annihilator of M.

We need to show $M_i = \ker(\alpha : M \to M_{P_i})$.

Consider the following commutative diagram:

$$\begin{tikzcd}
& M_{P_i} \arrow[dr, "\gamma"] & \\
M \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & (M/M_i)_{P_i} = M_{P_i}/(M_i)_{P_i} \\
& M/M_i \arrow[ur, "\delta"'] &
\end{tikzcd}$$

The kernel of β is M_i, so if we can show δ and γ

are injective, we're done.

Since M_i is P_i-primary, δ is an injection.

Since $\bigcap_j M_j = 0$, $\varphi : M \to \oplus M/M_j$ is injective.

$\Rightarrow \varphi_{P_i} : M_{P_i} \to \oplus (M/M_j)_{P_i}$ is injective.

Since P_i is minimal over ann M, $P_j \not\subset P_i$ for $i \neq j$,

But M/M_j is P_j-primary, so for $u \in P_j - P_i$, some power

of u annihilates M/M_j, so $(M/M_j)_{P_i} = 0$.

Thus γ is an injection as desired. $\square$

Primary decomposition and localization

Suppose we have a minimal decomp.

$$M' = \bigcap_{i=1}^{n} M_i \quad \text{for } M' \subseteq M, \quad \text{and } \{P_i\} \text{ the corr. primes.}$$

Let $U \subseteq R$ be multiplicative, and reindex the M_i and P_i s.t. $P_1, \dots, P_t$ are the P_i not meeting U.

Claim: $M'[U^{-1}] = \bigcap_{i=1}^{t} M_i[U^{-1}]$ is a minimal primary decomp. over $R[U^{-1}]$.

Pf: Again, by factoring out M', we can reduce to the case $M'=0$.

If $U \cap P_i = \emptyset$, then $\operatorname{Ass} M_i[U^{-1}] = \{P_i R[U^{-1}]\}$, so $M_i[U^{-1}]$ is $P_i R[U^{-1}]$-primary.

If $U \cap P_i \neq \emptyset$, then $(M/M_i)[U^{-1}] = 0 \Rightarrow M_i[U^{-1}] = M[U^{-1}]$.

Thus $\bigcap_{i=1}^{t} M_i[U^{-1}] = 0$ is a primary decomposition.

Since the associated primes of $M[U^{-1}]$ are those of M that don't meet U, this is minimal. □

Primary decomposition in UFDs

Recall that the motivation for primary decomposition was prime factorization in $\mathbb{Z}$, or in UFDs more generally. Now we show that the definitions agree.

Prop: Let R be a Noetherian integral domain.

a.) If $f \in R$ and $f = u \prod p_i^{e_i}$, s.t. u is a unit and the p_is are prime, generating distinct ideals, $e_i > 0$, then $(f) = \bigcap (p_i^{e_i})$ is the minimal primary decomposition of (f).

b.) R is a UFD $\Longleftrightarrow$ every prime ideal minimal over a principal ideal is principal.

Pf: a.) First we show $(p_i^{e_i})$ is (p_i)-primary.

Clearly a power of (p_i) annihilates $R/(p_i^{e_i})$.

If $r \in R-(p_i)$ and $\bar{m} \in R/(p_i^{e_i})$ s.t $r\bar{m}=0$, then $p_i^{e_i} \mid r\underset{\substack{\uparrow \\ \text{lift of} \\ \bar{m} \text{ in } R}}{m}$, but p_i doesn't divide r, so $\bar{m}=0$.

Clearly $(f) \subseteq \bigcap (p_i^{e_i})$, so we need to show the reverse inclusion.

By induction, we need to show $(g) \cap (p_1^{e_1}) \subseteq (f)$, where $g = u\prod_{i \neq 1} p_i^{e_i}$.

Let $r \in (g) \cap (p_1^{e_1})$. Then $r = gq \in (p_1^{e_1})$.
p_1 doesn't divide $g \Rightarrow p_1 \mid q$. Repeating this, $p_1^{e_1} \mid q \Rightarrow r \in (f)$, as desired.

b.) Suppose R is a UFD. If $f = u\prod_i p_i$ is the prime factorization of an element f, then by a.), the associated primes of $R/(f)$ are (p_i), so they are minimal over $\operatorname{ann}(R/(f)) = (f)$.

Conversely, suppose every prime ideal minimal over a principal ideal is principal.

If every irreducible element is prime, then factorizations are unique (and they exist by ACC). Thus, it suffices to show any irreducible $f \in R$ is prime.

If P is minimal over f, then $P = (p)$, so $f = pu$, some u. Since f is irreducible, u must be a unit, so $(f) = (p)$, so f is prime. $\square$

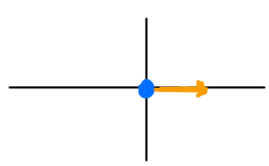
More geometric examples

We'll informally give some more geometric intuition behind the primary decomposition. Let $k = \bar{k}$.

Ex: Let $I = (x^2, y) \subseteq k[x,y]$. $V(I) = \{(x,y)\}$, so set theoretically this ideal corresponds to a point.

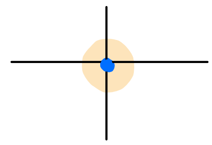
If $f = a_0 + a_1 x + a_2 y + a_3 x^2 + \ldots$ is a polynomial, then

its residue mod (x^2, y) preserves the scalars $a_0 = f(0,0)$ and $a_1 = \frac{\partial f}{\partial x}(0,0)$. i.e. it gives the value of the function at the origin and its derivative in the horizontal direction.

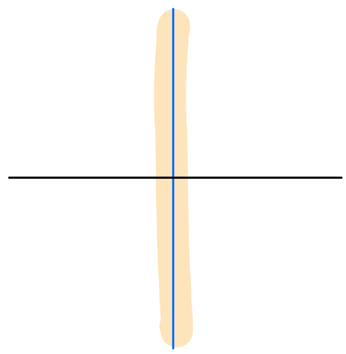


Also note: I contains (y) but not (x).

Ex: $I = (x^2, xy, y^2)$. Here if we look at f mod I, we get the value of f at 0 and the values of the first derivative in any direction. We can think of $V(I)$ as the whole first order infinitesimal neighborhood of the origin:



Ex: If $I = (x^2)$, the residue of f mod I tells the value of f at every point on the line $x = 0$, along w/ the values of its first derivatives in the horizontal direction.



From these ideas we're able to roughly understand geometrically what the primary decomposition means.

Ex: $I = (x^2, xy) = (x) \cap (x^2, xy, y^2)$ corresponds to the union of the vertical line, and a first order infinitesimal neighborhood of the origin. However, the only info about the first order behavior of a function that's not in the vertical line is in one other, e.g. the horizontal direction,

so $I = (x) \cap (x^2, y)$. Similarly,

$$((x-y)^2, (x+y)) \cap (x) = (x^2, xy)$$